Variational Methods for Computer Vision

Part 2: Bayesian Inference

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Segmentation and Prior Knowledge
Variational Image Segmentation

*Mumford, Shah ’89, Chambolle et al. ‘12:*

\[
\min_{\Omega_1,\ldots,\Omega_n} \sum_i \int_{\Omega_i} f_i \, dx + \nu |\partial \Omega_i|
\]

s.t. \( \bigcup_i \Omega_i = \Omega, \quad \Omega_i \cap \Omega_j = \emptyset \ \forall i \neq j \)

2 label segmentation

10 label segmentation
\[ C = \{ x \in \Omega \mid \phi(x) = 0 \}, \quad \phi : \Omega \subset \mathbb{R}^2 \to \mathbb{R} \]

*Dervieux, Thomasset ’79,’81, Osher, Sethian ’88*
Level Set Segmentation

Chan, Vese ’01
A Challenging Segmentation Problem

Cremers, IEEE PAMI '06
Statistical Interpretation

Given an image $I$, find the most likely segmentation by maximizing the conditional probability

$$P(C | I) = \frac{P(I | C) P(C)}{P(I)}$$

with respect to a curve $C$.

This is equivalent to minimizing its negative logarithm which (up to a constant) is given by the energy

$$E(C) = E_{data}(C) + E_{shape}(C)$$

with

$$E_{data} = -\log P(I | C) \text{ and } E_{shape} = -\log P(C)$$

Cremers, Osher, Soatto, IJCV ’06
Insufficient Low-Level Information
\[ P(\phi) \propto \frac{1}{N} \sum_{i=1}^{N} \exp \left( -\frac{1}{2\sigma^2} d^2(\phi, \phi_i) \right) \]

Cremers, Osher, Soatto, IJCV '06

Rosenblatt '56, Parzen '62
Statistical Shape Priors

\[ E_{data} + E_{length} \rightarrow \text{min} \]

\[ E_{data} + E_{shape} \rightarrow \text{min} \]

Cremers, Osher, Soatto, IJCV '06
Limitations of Static Shape Priors
Training sequence
Dynamical Models for Implicit Shapes

1. Low-dim. representation via PCA \((\text{Leventon et al. '00, Tsai et al. '01})\):

\[
\phi_i(x) \approx \phi_0(x) + \sum_{j=1}^{m} \alpha_{ij} \psi_j(x)
\]

\[
\alpha_{ij} = \int (\phi_i - \phi_0) \psi_j \, dx \\
\alpha_i = (\alpha_i^1, \ldots, \alpha_i^m)
\]

2. Autoregressive model for the shape coefficients:

\[
\alpha_t = \mu + \sum_{i=1}^{k} A_i \alpha_{t-i} + \eta
\]

mean transition matrices Gaussian noise

3. Synthesize shape vectors \(\alpha_t\) and embedding surfaces \(\phi_t\):

\[
\phi_t(x) = \phi_0(x) + \alpha_t^\top \psi(x)
\]
Dynamical Shape Priors

Statistically synthesized embedding functions

Cremers, IEEE PAMI 2006
1. Low-dim. representation via PCA (Leventon et al. ’00, Tsai et al. ’01):

\[
\phi_i(x) \approx \phi_0(x) + \sum_{j=1}^{m} \alpha_{ij} \psi_j(x)
\]

\[
\alpha_{ij} = \int (\phi_i - \phi_0) \psi_j \, dx \quad \alpha_i = (\alpha_{i1}, \ldots, \alpha_{im})
\]

2. Autoregressive model for the shape coefficients:

\[
\alpha_t = \mu + \sum_{i=1}^{k} A_i \alpha_{t-i} + \eta
\]

3. Synthesize shape vectors and embedding surfaces:

\[
\phi_t(x) = \phi_0(x) + \alpha_t^T \psi(x)
\]

Cremers, IEEE PAMI 2006
1. Low-dim. representation via PCA \((\text{Leventon et al. '00, Tsai et al. '01})\):

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\alpha_t = \mu + \sum_{i=1}^{k} A_i \alpha_{t-i} + \eta
\]

3. Synthesize shape vectors and embedding surfaces:

\[
\phi_t(x) = \phi_0(T_{\theta_t} x) + \alpha_t^\top \psi(T_{\theta_t} x)
\]

\textit{Cremers, IEEE PAMI 2006}
Dynamical Priors for Image Segmentation

Bayesian Aposteriori Maximization:

$$\hat{\alpha}_t, \hat{\theta}_t = \arg \max_{\alpha_t, \theta_t} \mathcal{P} \left( \alpha_t, \theta_t \mid I_t, \hat{\alpha}_{1:t-1}, \hat{\theta}_{1:t-1} \right)$$

$$= \arg \max_{\alpha_t, \theta_t} \mathcal{P} \left( I_t \mid \alpha_t, \theta_t \right) \mathcal{P} \left( \alpha_t \mid \hat{\alpha}_{1:t-1} \right)$$

$$E(\alpha_t, \theta_t) = -\log \mathcal{P} = E_{\text{dat}}(\alpha_t, \theta_t, I_t) + E_{\text{dyn}}(\alpha_t, \hat{\alpha}_{1:t-1})$$

$$E_{\text{dyn}} = \frac{1}{2} (\alpha_t - \nu) \top C^{-1} (\alpha_t - \nu), \quad \nu \equiv \mu + \sum_{i=1}^{k} A_i \hat{\alpha}_{t-i}$$

Optimization by gradient descent:

$$\frac{d\alpha_t}{d\tau} = -\frac{\partial E}{\partial \alpha_t}, \quad \frac{d\theta_t}{d\tau} = -\frac{\partial E}{\partial \theta_t}$$
Dynamical Shape Priors

Cremers, *IEEE PAMI '06*
Robustness to Speed Variation

2 x faster

3 x faster

4 x faster

5 x faster
3D Heart Tracking with Dynamical Prior

4d Shape Priors Using Auto-Regressive Models on PCA-Subspaces

Implementation: Timo Kohlberger
3D Heart Tracking with Dynamical Prior

Implementation: Timo Kohlberger
Dynamical shape prior

Cremers, *IEEE PAMI ’06*
Dynamical prior of \textit{shape} and \textit{translation}

\textit{Cremers, IEEE PAMI '06}
Summary

Bayesian inference

Statistical shape priors

Dynamical shape priors

3D heart tracking