Variational Methods for Computer Vision

Part 1: Variational Calculus

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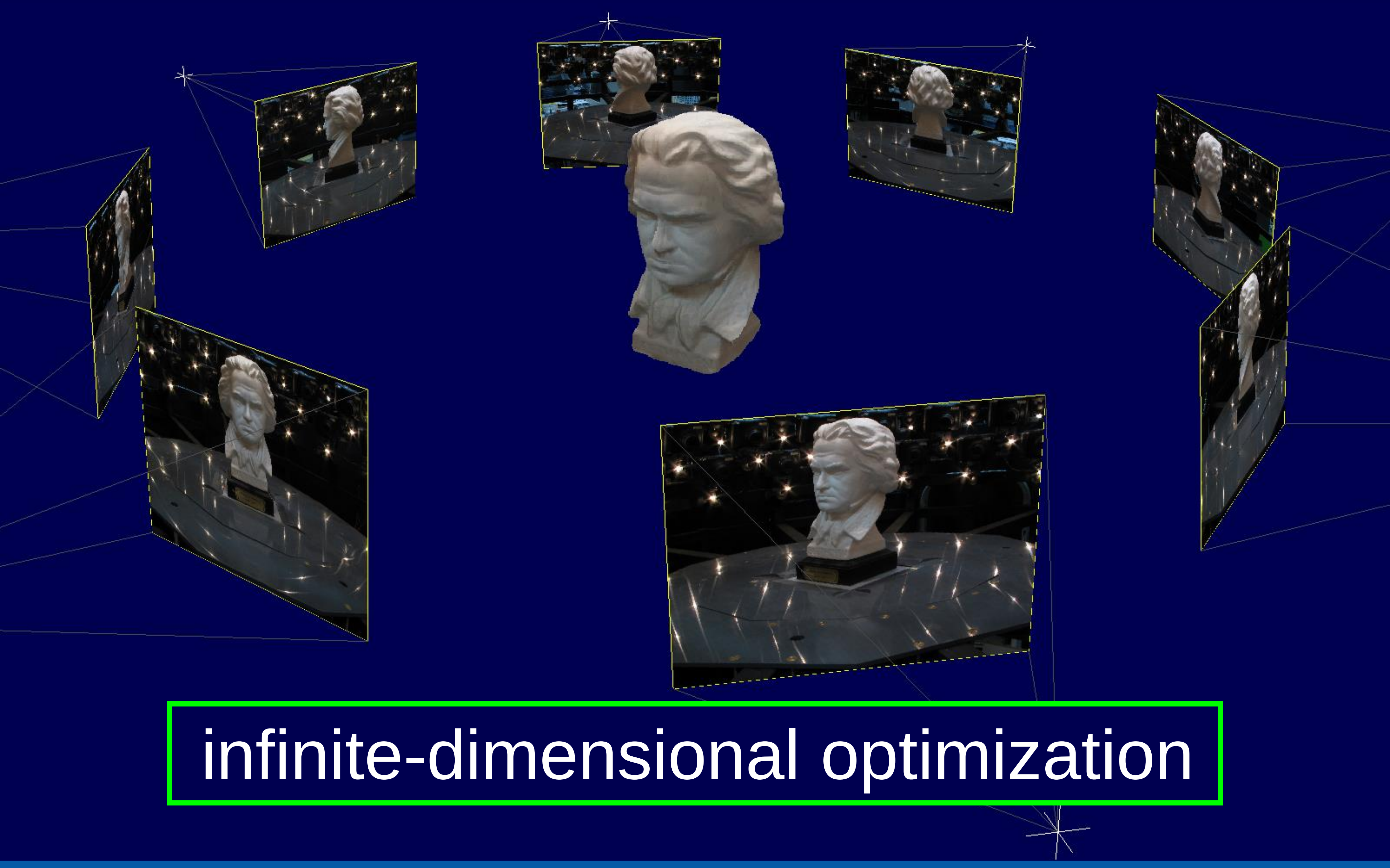
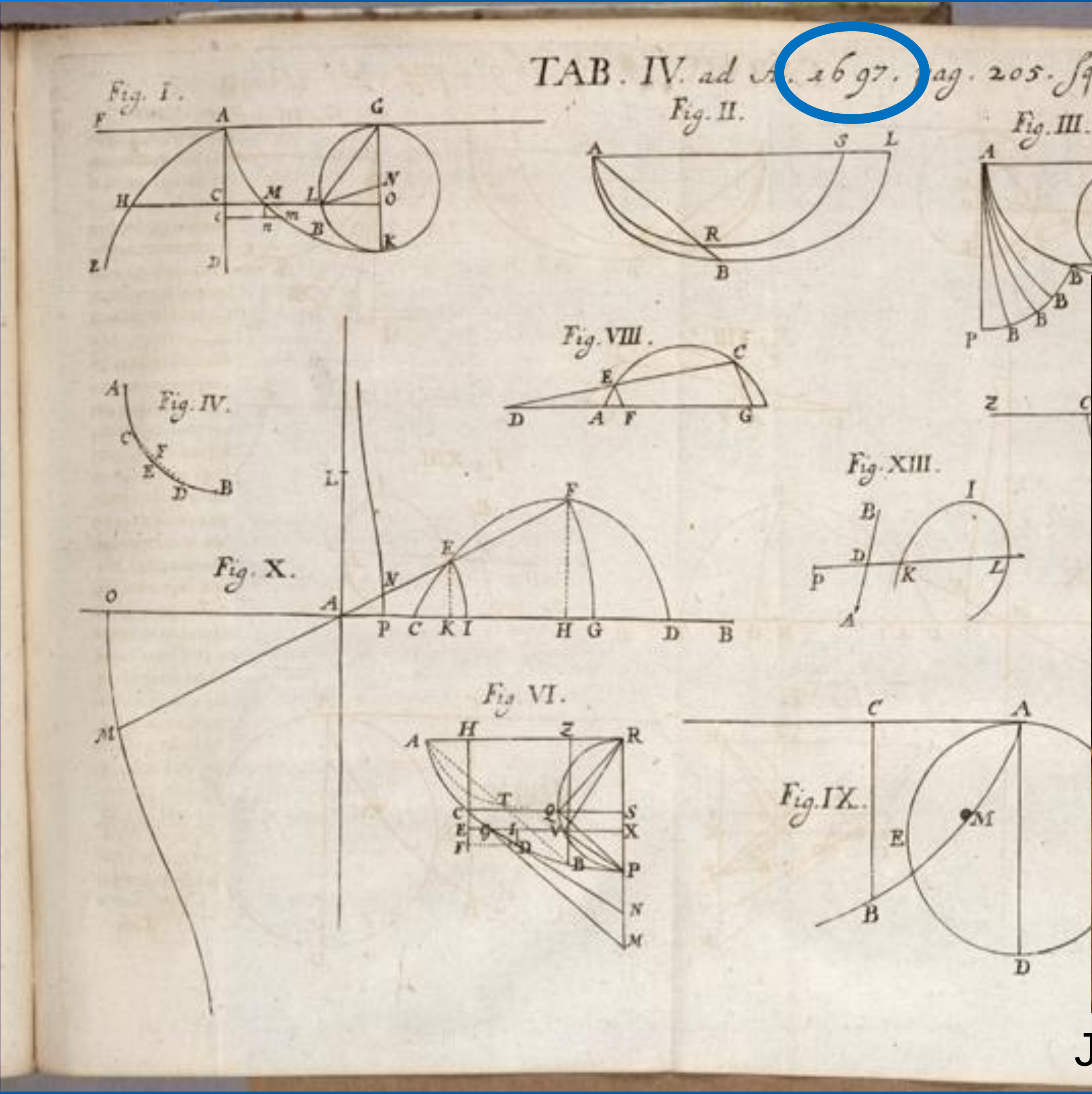
Spatially Dense Reconstruction

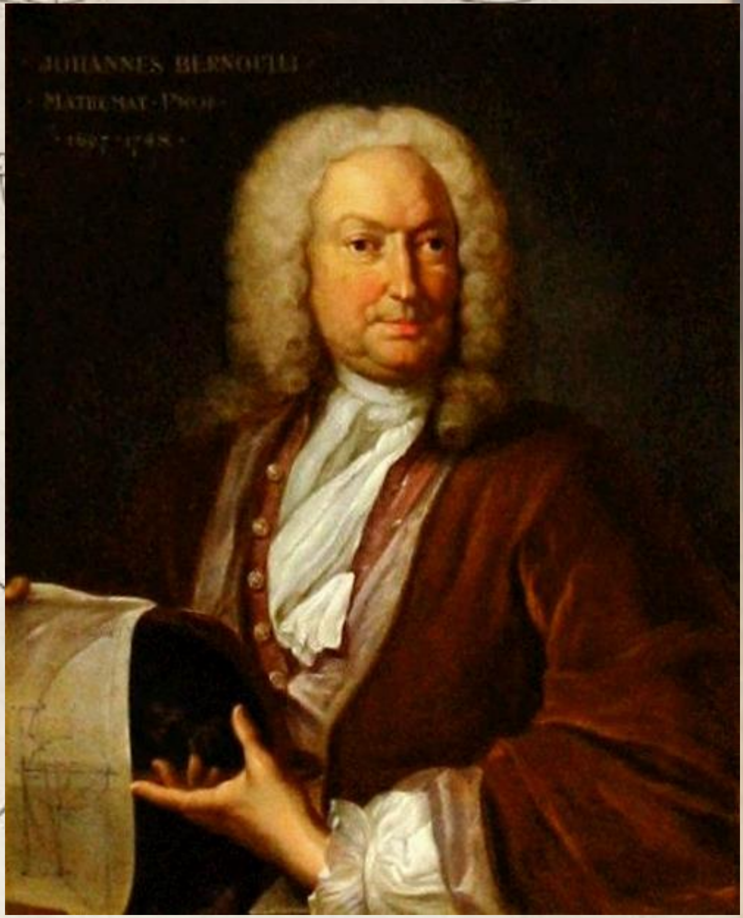


infinite-dimensional optimization

Which path is the fastest?

Bernoulli & The Brachistochrone





Johann Bernoulli (1667-1748)

Image segmentation:

Kass et al. ’88, Mumford, Shah ’89, Caselles et al. ‘95,

Kichenassamy et al. ‘95, Paragios, Deriche ’99,

Chan, Vese ‘01, Tsai et al. ‘01, …

Multiview stereo reconstruction:

Faugeras, Keriven ’98, Duan et al. ‘04, Yezzi, Soatto ‘03,

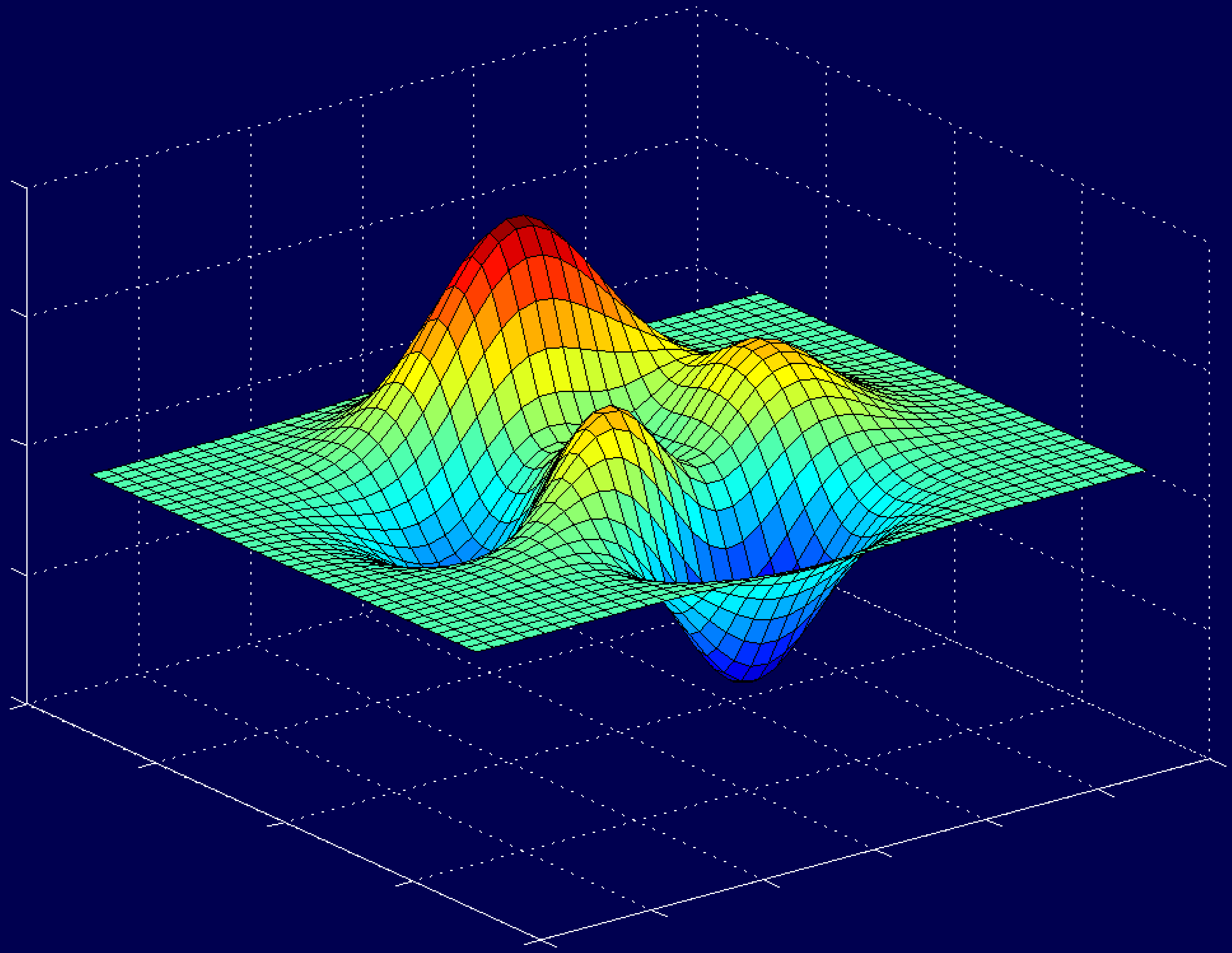
Labatut et al. ’07, Kolev et al. IJCV ‘09…

Optical flow estimation:

Horn, Schunck ‘81, Nagel, Enkelmann ‘86, Black, Anandan ‘93,

Alvarez et al. ‘99, Brox et al. ‘04, Zach et al. ‘07, Sun et al. ‘08,

Wedel et al. ’09, Werlberger et al. ‘10, …



Compute solutions via energy minimization

For $u : \Omega \to \mathbb{R}, \ \Omega \subset \mathbb{R}^2,$ compute:

$$u_{den} = \arg\min_u \int_\Omega (u - f)^2 \, dx + \lambda \int_\Omega |\nabla u| \, dx$$

input image $f : \Omega \to \mathbb{R}^3$

denoised $u_{den} : \Omega \to \mathbb{R}^3$

Rudin, Osher, Fatemi, Physica D '92

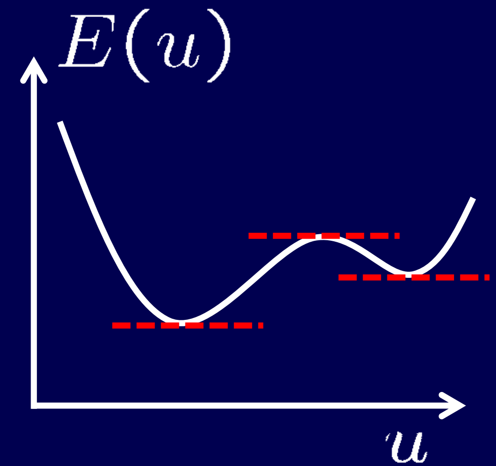
Variational Methods & PDEs

For $u : \Omega \to \mathbb{R},\ \Omega \subset \mathbb{R}^2,$ consider the functional

$$E(u) = \int_\Omega (u - f)^2 \, dx + \lambda \int_\Omega |\nabla u|^2 \, dx$$

Extremality principle as a necessary condition:

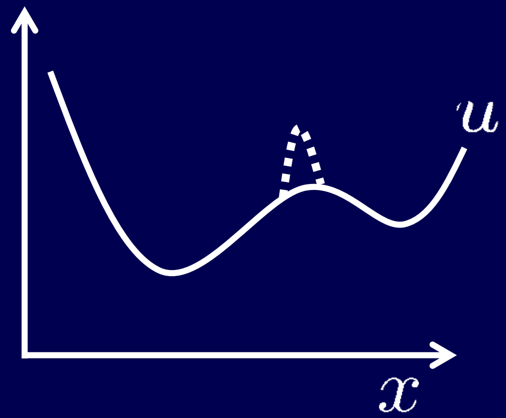
$$\frac{dE}{du} = 0$$



But: How to define the derivative with respect to a function?

$$E(u) = \int_\Omega (u - f)^2 + \lambda \, |u'|^2 \, dx = \int_\Omega \mathcal{L}(u, u') \, dx$$

Derivative in “direction” h :



$$\left.\frac{dE}{du}\right|_h = \lim_{\epsilon \to 0} \frac{E(u + \epsilon h) - E(u)}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{\int \mathcal{L}(u + \epsilon h, u' + \epsilon h') - \mathcal{L}(u, u') \, dx}{\epsilon}$$

Taylor expansion:

$$= \lim_{\epsilon \to 0} \frac{\int \cancel{\mathcal{L}(u, u')} + \cancel{\epsilon} h \frac{\partial \mathcal{L}}{\partial u} + \cancel{\epsilon} h' \frac{\partial \mathcal{L}}{\partial u'} - \cancel{\mathcal{L}(u, u')} \, dx}{\cancel{\epsilon}}$$

The Gateaux Derivative

$$\left.\frac{dE}{du}\right|_h = \int h \frac{\partial \mathcal{L}}{\partial u} + h' \frac{\partial \mathcal{L}}{\partial u'}\, dx$$

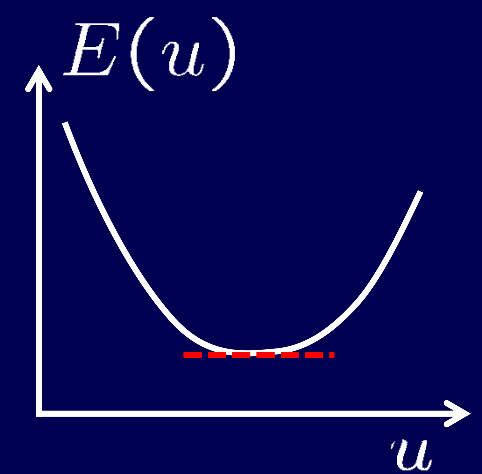
Integration by parts:

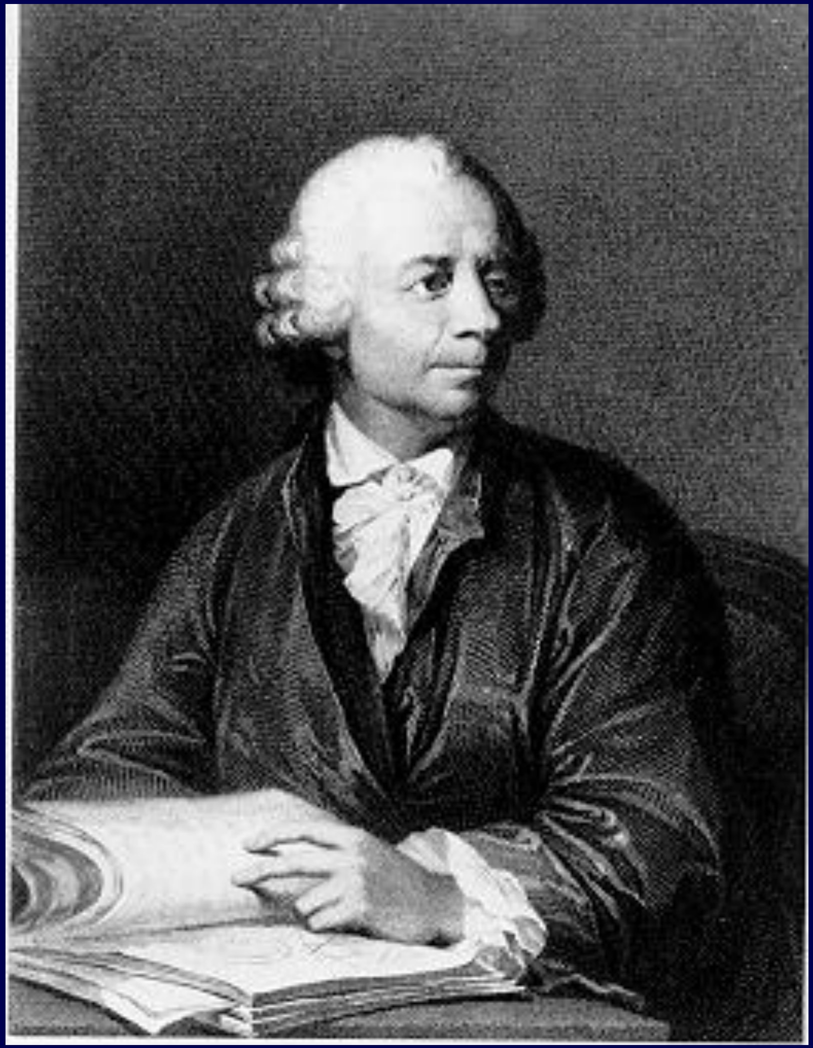
$$= \int h \left(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u'} \right) \right) dx \stackrel{!}{=} \int h \, \frac{dE}{du}\, dx$$

Necessary optimality condition:

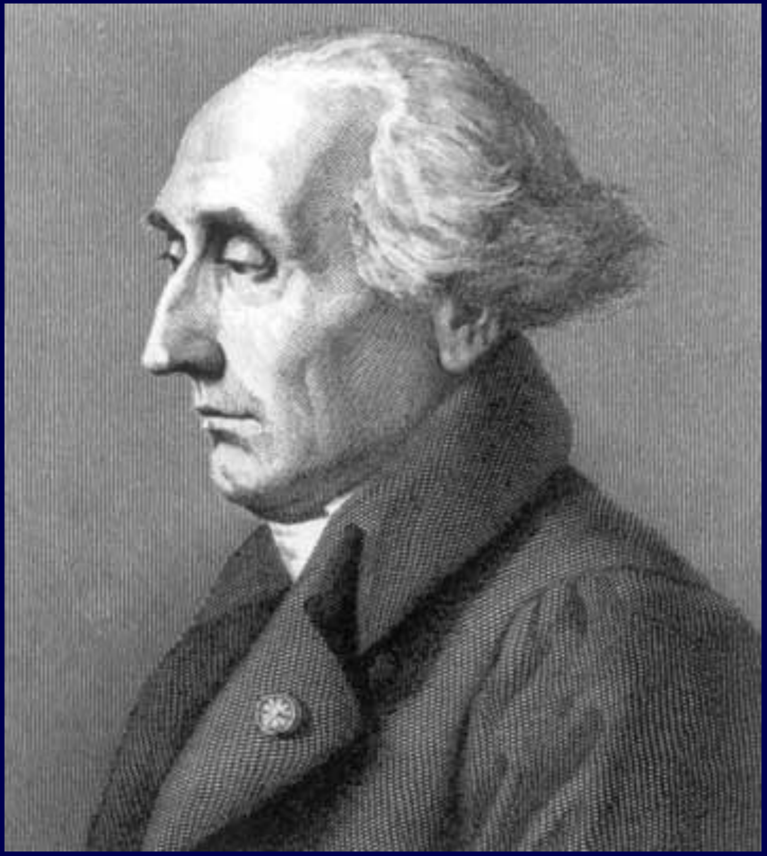
$$\frac{dE}{du} = \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u'} \right) = 0$$

Euler-Lagrange equation





Leonhard Euler
(1703-1783)

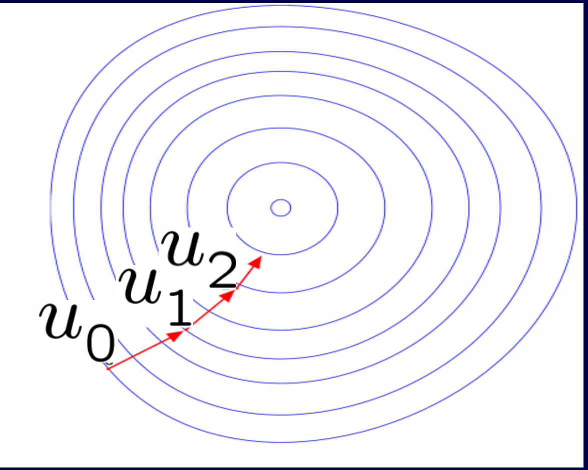


Joseph-Louis Lagrange
(1736 – 1813)

Iteratively walk "down-hill":

$$\begin{cases} u(x,0) = u_o(x) \\ \frac{\partial u}{\partial t} = -\frac{dE}{du} = -\frac{\partial \mathcal{L}}{\partial u} + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u'}\right) \end{cases}$$

direction of steepest descent



For the energy

$$E(u) = \frac{1}{2}\int_\Omega (u-f)^2 + \lambda\,|u'|^2\,dx$$

we obtain:

$$\frac{\partial u}{\partial t} = f - u + \lambda u''$$

diffusion equation

$$\int_\Omega |\nabla u|^2 \, dx \;\rightarrow\; \text{min}$$

Slightly modify the regularization (*Rudin, Osher, Fatemi ‘92*):

$$E(u) = \frac{1}{2}\int_\Omega (u-f)^2\,dx + \frac{\lambda}{2}\int_\Omega |\nabla u|^{\not 2}\,dx$$

total variation

Gradient descent:

$$\begin{cases} u(x,0) = u_o(x) \\ \dfrac{\partial u}{\partial t} = -\dfrac{dE}{du} = f - u + \lambda\,\text{div}\,(g\,\nabla u)\,, \qquad g = \dfrac{1}{|\nabla u|} \end{cases}$$

nonlinear diffusion

Perona, Malik ‘90, Rudin et al. ’92, Weickert ‘98

Linear vs. Nonlinear Diffusion

$$\int_\Omega |\nabla u|^2 \, dx \;\rightarrow\; \min$$

$$\int_\Omega |\nabla u| \, dx \;\rightarrow\; \min$$

The expression

$$\int |\nabla u| \, dx$$

only applies to differentiable functions. It is also non-differentiable.

Using the identity $|\nabla u| = \sup_{|\xi| \leq 1} \xi \cdot \nabla u$, we define the total variation

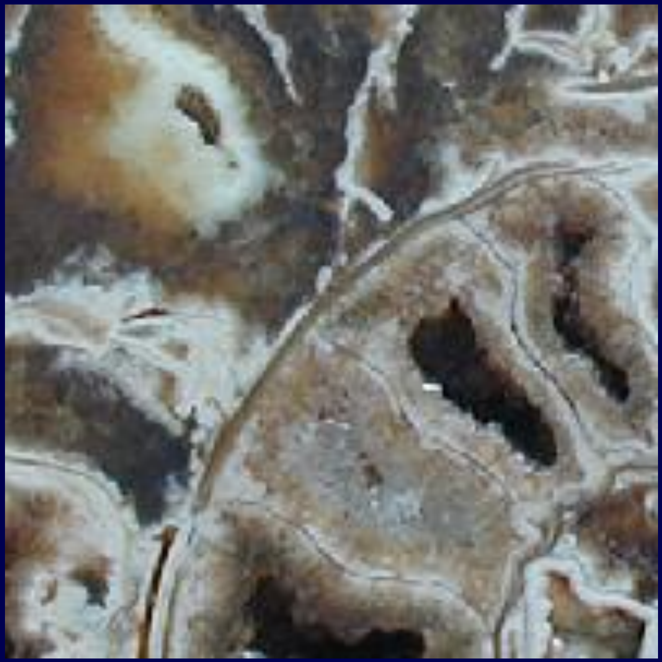
$$\mathrm{TV}(u) := \sup_{\xi \in \mathcal{K}} \int u \, \mathrm{div}\xi \, dx$$

with

$$\mathcal{K} = \left\{ \xi \in C_c^1(\Omega; \mathbb{R}^2) \,\middle|\, |\xi(x)| \leq 1 \; \forall x \in \Omega \right\}.$$

Variational Deblurring

$$u_{deb} = \arg\min_u \int_\Omega (b * u - f)^2 \, dx + \lambda \int_\Omega |\nabla u| \, dx$$



original image

blurred image f

deblurred image u_{deb}

Lions, Osher, Rudin, '92

Variational Super-Resolution

$$u_{sr} = \arg\min_u \sum_i \|D_i u - f_i\| + \lambda \int_\Omega |\nabla u| \, dx$$

One of several input images f_i

Super-resolution estimate u_{sr}

Schoenemann, Cremers, IEEE TIP '12

Variational Optical Flow

$$\min_{u:\Omega\to\mathbb{R}^2} \int_\Omega \left| f_1(x) - f_2(x+u) \right| dx \; + \; J(u)$$

Input video

Optical flow field

Horn & Schunck ‘81, Zach et al. DAGM ’07, Wedel et al. ICCV ’09

$$\min_{u:\Omega\to\mathbb{R}^2} \int_\Omega \left| f_1(x) - f_2(x+u) \right| dx + J(u)$$

Input video

Optical flow field*

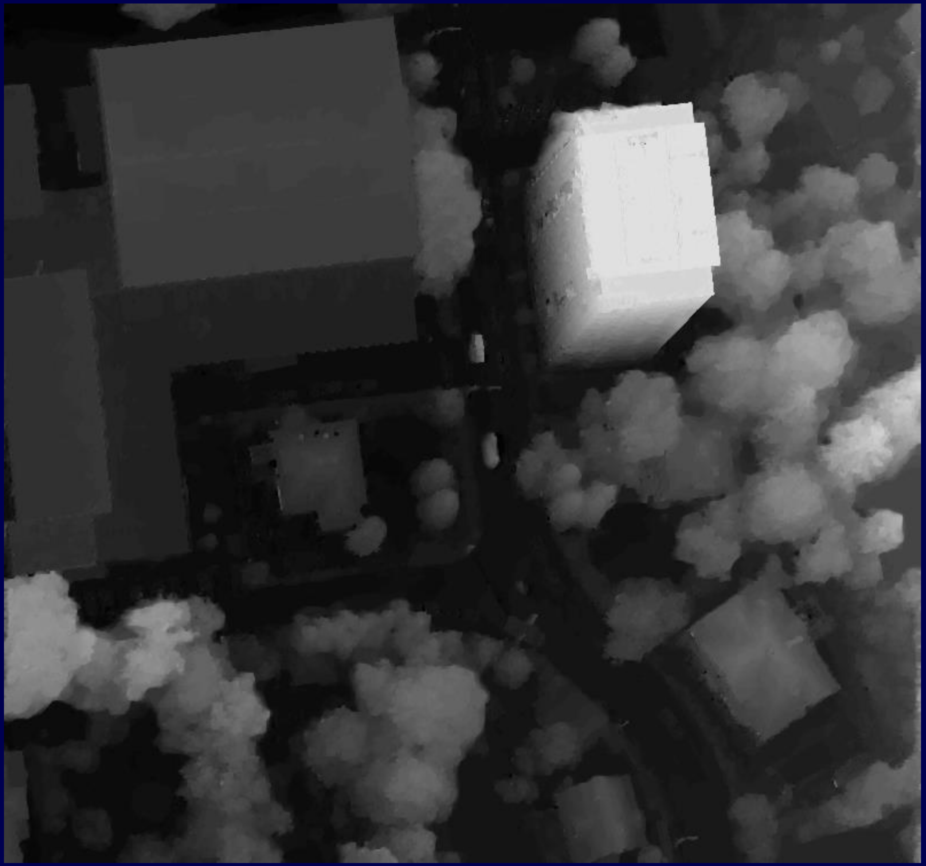
* 60 fps @ 640x480

Horn & Schunck '81, Zach et al. DAGM '07, Wedel et al. ICCV '09

Variational Stereo Reconstruction

$$\min_{u:\Omega\to\mathbb{R}} \int_\Omega \rho(x,u)\,dx + J(u)$$

one of two input images

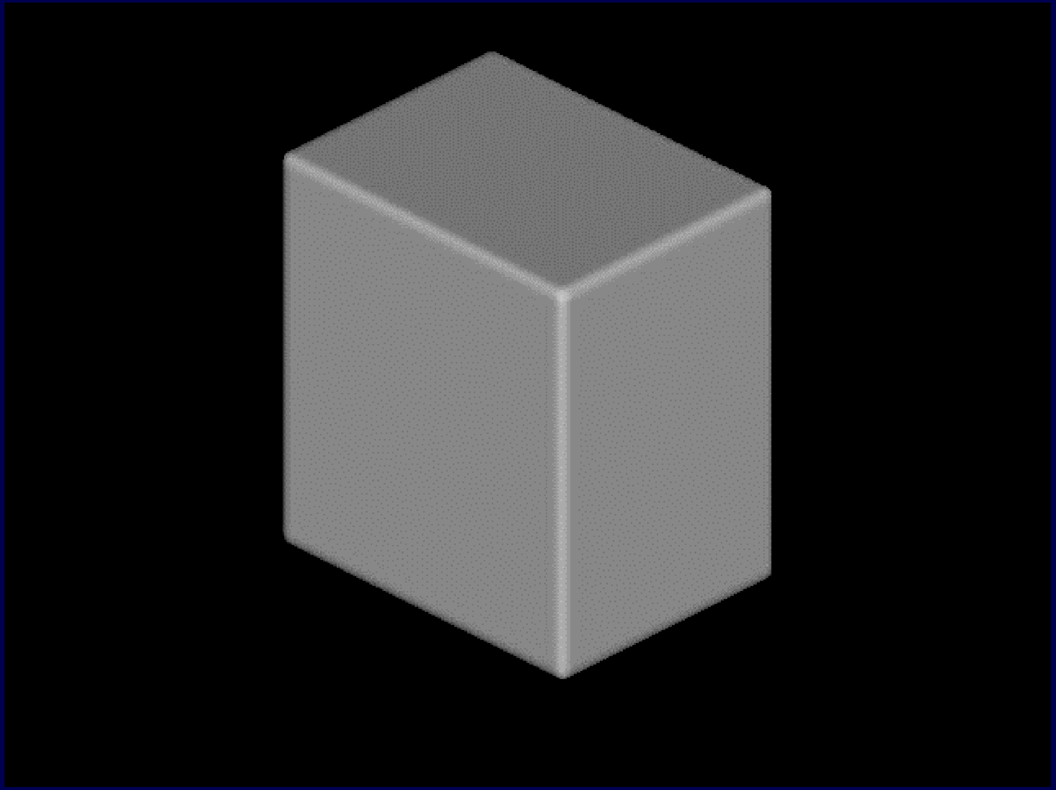


depth reconstruction

Pock, Schoenemann, Graber, Bischof, Cremers ECCV '08

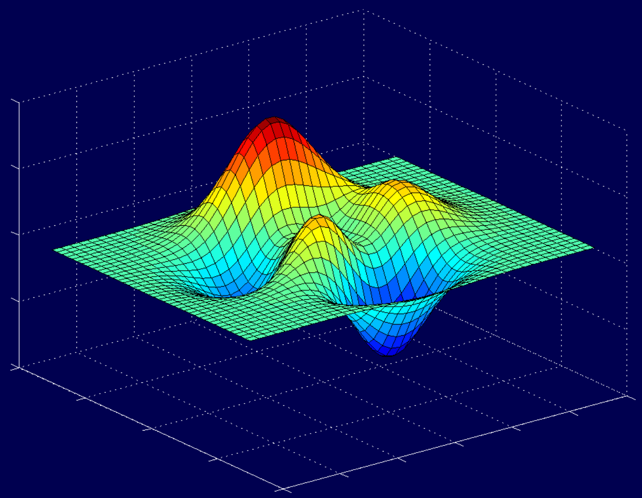
Variational Scene Flow

Wedel & Cremers, "Scene Flow", Springer 2011



Kolev et al., IJCV '09

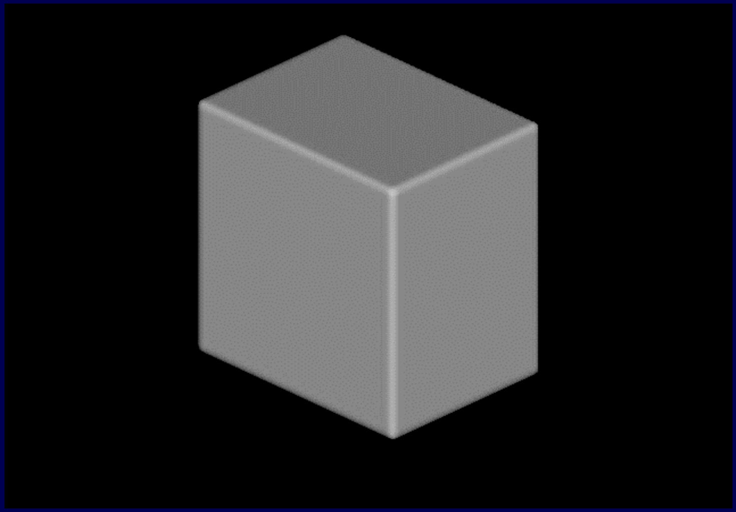
Summary



variational methods

optical flow

scene flow



multiview reconstruction